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Universal Multifractal Properties of Circle Maps from the Point of View of Critical Phenomena. II. Analytical Results

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The multifractal properties of maps of the circle exhibited in the preceding paper are analyzed from a simplified approach to the renormalization group of Kadanoff. This "second" renormalization group transformation, whose formulation and interpretation are discussed here, acts on the space of one-time-differentiable coordinate changes which associate a map on the critical manifold to the fixed point of the usual renormalization group. While the dependence of the multifractal moments on the starting point can be described statistically, and in particular through universal amplitude ratios as in paper I, it is shown that Fourier analysis is another possible approach. For all multifractal moments, the low-frequency Fourier coefficients have a universal self-similar scaling behavior analogous to that found for the usual spectrum of circle maps. In the case of the first moment, it is demonstrated that the Fourier coefficients are, within constants, equal to the usual spectrum. The relation between amplitude ratios and Fourier coefficients is established and it is demonstrated that the universal values of the ratios come from the universal low-frequency Fourier coefficients. Since, for the universal ratios arising in the statistical description, the scaling regime is much more easily accessible than for the spectrum, the statistical approach described in paper I should be more convenient for experiments and could become an alternative to the usual spectral description. The universal statistical description of the multifractal moments adopted here is possible because the choice of the *a priori* probability for the starting point is demonstrated to be irrelevant.

KEY WORDS: Onset of chaos; circle map; quasiperiodic route to chaos; multifractals; infinite set of exponents; critical phenomena; universality.

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1. INTRODUCTION

In the preceding paper (referred to as I), it has been shown that the multifractal properties of maps of the circle at and near criticality can be formulated in a way which closely parallels the phenomenology of critical phenomena. This approach offers the advantage of focusing on not only the exponents, but also the analog of universal amplitude ratios, which should be accessible from the same experiments which extract exponents. The purpose of this second part is to provide analytical arguments as well as interpretations for some of the results of I. There already exists a renormalization group (RG) approach, due to Kadanoff,[1] for the multifractal properties of maps of the circle[2] at the golden-mean rotation number. The main contribution of the present approach is to simplify the formalism of Kadanoff[1] and to provide further understanding of the universal nature of the multifractal properties and of their relation to the usual RG for maps of the circle.[3–5] While the universality of multifractal exponents folows from the analysis of Kadanoff, here it is further shown, using Fourier analysis, that the universal value of the ratios defined in I is concomitant with a scaling law analogous to the one characterizing the spectrum at the onset of chaos. The point of this study of Fourier coefficients is to demonstrate that, when the multifractal amplitudes introduced in I are taken into account, the multifractal and spectral analyses are two alternate ways to characterize the universal properties of the trajectory. In the case of the $q=1$ multifractal moment, this connection is clearly established and the universal character of the amplitude ratios is demonstrated to derive from the low-frequency limit of the spectrum.

Section 2 recalls the statement of the problem in terms of the homeomorphism which relates, by a coordinate change, a map to a pure rotation with the same rotation number. For a mean rotation number equal to the golden mean, Section 3 formulates the "second" RG[1] characterizing the multifractal properties of the trajectory. This "second" RG analysis relies on the formulation of Rand *et al.*[3,4] for the RG describing the transition to chaos via the quasiperiodic route. As a first illustration, it is explicitly shown how the linear approximation of Rand *et al.* to the fixed-point function leads to a very accurate formula for the multifractal exponents $\tau(q)$. Finally, in Section 4, this renormalilzation group analysis is shown to imply that the ratios computed in the preceding paper are indeed universal. Although little is rigorously proved in the general case of an arbitrary multifractal moment, our Fourier analysis exhibits the similarities with the properties of the spectrum of maps of the circle at the onset of chaos. As in the latter case, the Fourier coefficients of the multifractal moments, considered as a function of the starting point of the series, are self-similar in the low-frequency limit and the universal values of the ratios introduced in I follow from the low-frequency universality. In that limit, they obey a fixed-point equation which can be naturally derived using the results of Section 3. We present a numerical analysis to support these results. In the case of the $q=1$ multifractal moment, the analytical analysis can be pushed further to show explicitly that the Fourier coefficients of the $q=1$ moment are a simple function of the spectrum. Further discussion of the significance of the RG for multifractal properties and its relation to the usual RG analysis may be found in the conclusion.

Two appendices contain detailed proofs. The first one shows explicitly how the averages of multifractal moments with respect to the starting point are independent of the *a priori* probability distribution. In the second appendix, it is demonstrated that the *averaged* multifractal moments are related to the conjugacy to a pure rotation, a result which was implicitly assumed in ref. 1. In the sequel, we follow the notations of Rand *et al.*[3,4] for the quasiperiodic route to chaos. A short version of this paper has appeared, and an early version of the complete work is contained in a Ph.D. thesis.[6]

2. STATEMENT OF THE PROBLEM: THE CONJUGACY TO A PURE ROTATION

Given a map f of the circle onto itself, let h be the conjugate homeomorphism to a pure rotation with the same rotation number [Eq. (20) and (21) of paper I or Eq. (4) below]. The purpose of this section is to show that the problem of the multifractal moments can be stated in terms of this change of coordinate and, as a result, the average multifractal moments can be expressed as moments of finite-difference slopes of h. Although h is not differentiable at the critical point, it is known[3-5] that it obeys two functional fixed-point identities on its domain of definition. From these fixed-point equations, it is possible to recover the renormalization group formalism of Kadanoff for the multifractal properties.

Let us first consider this coordinate change in terms of the original map of the circle. For our purpose, it suffices to know that the conjugacy of a circle map f to a pure rotation can be obtained (see ref. 7 for a discussion on this point) by considering the n-infinite limit for the series of functions (ρ is the mean winding number as in I),

$$g_n(x) \equiv \frac{1}{n} \sum_{k=1}^{n} [f^{(k)}(x) - k\rho] \tag{1}$$

which gives the inverse homeomorphism[7,8] h^{-1} (as in I) as $n \to \infty$. In other words, if we let

$$h^{-1} \equiv g \equiv \lim_{n \to \infty} g_n(x) \tag{2}$$

then

$$g \circ f = R_\rho \circ g \tag{3}$$

or, with $g \equiv h^{-1}$,

$$f \circ h = h \circ R_\rho \qquad \text{or} \qquad f = h \circ R_\rho \circ h^{-1} \tag{4}$$

where R_ρ is a pure rotation with rotation number ρ. Thus, Eq. (1) defines a change of coordinate to a pure rotation with the same rotation number.

Because of (1), the moments of the closest-return distrances, $M_q(F_n, x_1)$, defined by Eq. (10) of I, are related to h by

$$\langle\!\langle M_q(F_n, x_1) \rangle\!\rangle = \frac{1}{F_{n+1}} \left\langle\!\!\left\langle \sum_{1 \leqslant i \leqslant F_{n+1}} |\hat{f}^{(F_n)}(x_i) - x_i|^q \right\rangle\!\!\right\rangle \tag{5a}$$

$$\cong \int_{-\sigma^2}^{\sigma} |h(u - (-\sigma)^n) - h(u)|^q \, du \tag{5b}$$

where the brackets refer to an average over the starting point x_1. In Appendix A, it is shown that the result (5a) is independent of the *a priori* probability distribution for the starting of the sequence. A proof of Eq. (5b) and of Eq. (1) is presented in Appendix B. An alternate heuristic proof of Eq. (5) would follow the steps of Eqs. (25)–(29) below.

We can now give a simple picture for the multifractal moments in terms of the homeomorphism h. As n goes to infinity, σ^n tends to zero and Eq. (5) can be interpreted as the average, over the interval $[-\sigma^2, \sigma]$, of powers of the finite-difference slopes of $h(u)$. Indeed, as noted by Kadanoff,[1] the scaling properties of Eq. (5) are related to differential properties of h. To illustrate this point, let us assume that the first-order approximation

$$h(u - (-\sigma)^n) - h(u) \cong -(-\sigma)^n \, dh/du \tag{6}$$

holds everywhere, but at a finite number of points. The scaling properties of the moments are then obtained from

$$\int_{-\sigma^2}^{\sigma} |h(u - (-\sigma)^n) - h(u)|^q \, du \cong \sigma^{nq} \int_{-\sigma}^{\sigma} |dh/du|^q \, du \sim F_{n+1}^{-\tau(q)-1} \tag{7}$$

which gives a linear dependence $\tau(q) = q - 1$ for the exponents $[F_{n+1}^{-1} \cong \sigma^n(1+\sigma^2)]$. This contradicts the numerical results at the critical point where $\tau(q)$ is not a linear function of q. This is understood as follows. From an operational point of view, h can be easily obtained numerically by using the series defined in (1). As discussed in refs. 3–5 and 7, h is a *continuous* increasing function. However, the hypothesis of differentiability is not numerically corroborated at the critical point,[3–5] as is illustrated in Fig. 1 for the sine map [Eq. (1a) of I]. This is the reason for the failure of the expansion (6). This is why further progress at the critical point must rely instead on an RG approach.

In the sequel, we shall use the functional fixed-point equations for the homeomorphism which conjugates the universal function f_* of the fixed point to a pure rotation. Since h sends $[-\sigma^2, \sigma]$ onto an interval of length

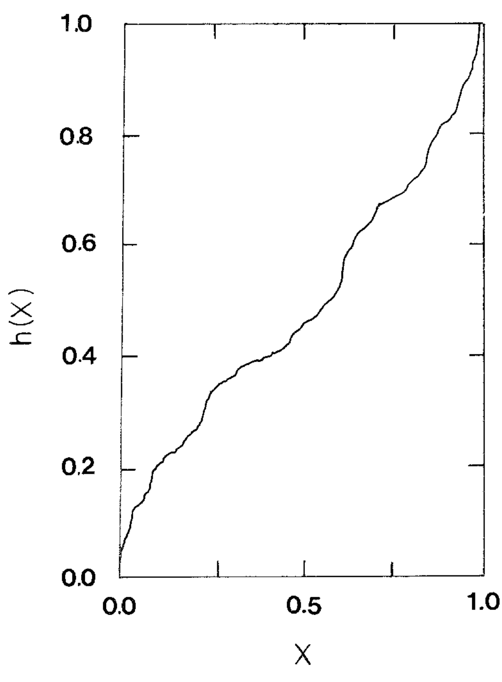

Fig. 1. Plot of the homeomorphism $h(x) = g^{-1}(x)$ as defined by Eq. (1) when the control parameters are taken at their critical values for the standard map defined in Eq. (1a) of I. As explained in the text, the multifractal moment of order q is the average qth moment of the *coarse-grained* slope of this function at scale σ^n {i.e., the average of $|h(x-(-\sigma)^n) - h(x)|^q \sigma^{-nq}$ over x on an interval of length unity}. If this function were differentiable, the multifractal exponents would be a trite linear function of the order of the moment. Following refs. 3–5, this function is, however, nowhere differentiable, as can be guessed from the rapid fluctuations of the slope.

1, we can define, in conformity with the notation of Rand *et al.*,[3,4] the domain of the map f_* as $[(\alpha-1)^{-1}, \alpha(\alpha-1)^{-1}]$, where α is the universal constant found by Shenker[5] ($\alpha = -1.28...$). In refs. 3 and 4, then, it is shown that h_* satisfies the following fixed-point equations:

$$h_*(\theta) = \begin{cases} \alpha^{-1} h_*\left(\dfrac{-\theta}{\sigma}\right) & \text{if } -\sigma^2 < \theta \leqslant \sigma^3;\ h_*(\theta) \in \left[\dfrac{1}{\alpha-1}, \dfrac{\alpha^{-1}}{\alpha-1}\right] \quad \text{(8a)} \\ f_*^{-1}\left(\alpha^{-2} h_*\left[\dfrac{\theta}{\sigma^2-1}\right]\right) & \text{if } \sigma^3 \leqslant \theta \leqslant \sigma;\ h_*(\theta) \in \left[\dfrac{\alpha^{-1}}{\alpha-1}, \dfrac{\alpha}{\alpha-1}\right] \quad \text{(8b)} \end{cases}$$

From these equations, a "second" RG is derived below. The set $\tau(q)$ comes from the "dominant" eigenvalues of this "second" RG. We use this terminology because the properties of the second RG are controlled by those of the usual (first) one [Eqs. (8)], whereas none of the operators of the first RG depend on, or are modified by, those of the second.

3. THE RENORMALIZATION GROUP

The starting point of the following approach is the coordinate change between a critical map of the circle (i.e., a map on the critical manifold) and the fixed point f_* of the renormalization group. By definition, this coordinate change relates f to f_* as follows:

$$f = \hbar \circ f_* \circ \hbar^{-1} \tag{9}$$

where Rand[9],[4] has recently proved that $\hbar$ is at least one-time differentiable when f is on the critical manifold of the golden-mean fixed point. By using the definition $f_* \circ h_* = h_* \circ R_\sigma$ of the fixed-point homeomorphism, one obtains

$$f \circ \hbar \circ h_* = \hbar \circ h_* \circ R_\sigma \tag{10}$$

As a consequence, the homeomorphism which conjugates f to a pure rotation is $\hbar \circ h_*$. From Eq. (5), the average multifractal moment can now be rewritten as

$$\int_{-\sigma^2}^{\sigma} |\hbar \circ h_*(u - (-\sigma)^n) - \hbar \circ h_*(u)|^q \, du \tag{11}$$

[4] The invariance of the $\tau(q)$ with respect to homeomorphic conjugacies which are lipschitz continuouos as also been discussed by A. Arneodo and M. Holschneider.[10] If this conjugacy is not of that type, these authors show that the maps are not in the same universality class.

and the integrand can be approximated by expanding $\hbar$ to linear order, since $\hbar$ is differentiable

$$\int_{-\sigma^2}^{\sigma} |h_*(u-(-\sigma)^n)-h_*(u)|^q\, |\hbar'(h_*(u))|^q\, du \tag{12}$$

To construct a renormalization group in the space of the at least one-time differentiable changes of coordinate $\hbar$, the functional fixed-point equations (8) will be used to relate the functional (12) at different time scales (i.e., different n).

To derive this scaling relationship, it suffices to split the integral in Eq. (12) in the two intervals $[-\sigma^2, \sigma^3]$ and $[\sigma^3, \sigma]$. For the first one, we change the variable to $v = -u/\sigma$ and by using Eq. (8), we obtain

$$\begin{aligned}\int_{-\sigma^2}^{\sigma^3} &|h_*(u-(-\sigma)^n)-h_*(u)|^q\, |\hbar'(h_*(u))|^q\, du \\ &= \sigma\, |\alpha|^{-q} \int_{-\sigma^2}^{\sigma} |h_*(v-(-\sigma)^{n-1})-h_*(v)|^q\, |\hbar'(h_*(v)/\alpha)|^q\, dv \end{aligned}\tag{13}$$

Actually, Eq. (13) defines a functional transformation, so that the right-hand side can be rewritten as

$$\sigma \int_{-\sigma^2}^{\sigma} |h_*(v-(-\sigma)^{n-1})-h_*(v)|^q\, |\mathscr{R}_1[\hbar](h_*(v))|^q\, dv \tag{14}$$

where $\mathscr{R}_1[\hbar](x)$ is a function of argument x, defined by

$$\mathscr{R}_1[\hbar](x) \equiv \frac{d}{dx}\left[\hbar\left(\frac{x}{\alpha}\right)\right]; \qquad x \in [(\alpha-1)^{-1}, \alpha(\alpha-1)^{-1}] \tag{15}$$

For the second integral, the change of variable $v = u/\sigma^2 - 1$ leads to

$$\begin{aligned}\int_{\sigma^3}^{\sigma} &|h_*(u-(-\sigma)^n)-h_*(u)|^q\, |\hbar'(h_*(u))|^q\, du \\ &= \sigma^2 \int_{-\sigma^2}^{\sigma} dv\, \{|h_*(\sigma^2(v+1)-(-\sigma)^n)-h_*(\sigma^2(v+1))|^q \\ &\quad \times |\hbar'\{h_*(\sigma^2(v+1))\}|^q\} \end{aligned}\tag{16}$$

In order to get a second scaling relationship, it suffices to use (8b) to obtain

$$h_*\{\sigma^2[v+1]-(-\sigma)^n\}-h_*\{\sigma^2(v+1)\}$$
$$=f_*^{-1}(\alpha^{-2}h_*\{v-(-\sigma)^{n-2}\})-f_*^{-1}(\alpha^{-2}\hbar_*(v))$$
$$\cong (h_*\{v-(-\sigma)^{n-2}\}-h_*(v))\times\alpha^{-2}\, df_*^{-1}/dx|_{x=\alpha^{-2}h_*(v)} \tag{17}$$

the last step being justified by the fact[1] that the derivative of f_*^{-1} is bounded on the interval considered in Eq. (17), since it does not include the origin.

By using the chain rule for the derivative, Eq. (17) can now be written as

$$\int_{\sigma^3}^{\sigma} |h_*(u-(-\sigma)^n)-h_*(u)|^q\, |\hbar'(h_*(u))|^q\, du$$
$$=\sigma^2\int_{-\sigma^2}^{\sigma} |h_*(v-(-\sigma)^{n-2})-h_*(v)|^q\, |\mathscr{R}_2[\hbar](h_*(v))|^q\, dv \tag{18}$$

where the second functional transformation $\mathscr{R}_2$ is defined by

$$\mathscr{R}_2[\hbar](x)=\frac{d}{dx}[\hbar\circ f_*^{-1}\circ\alpha^{-2}](x);\qquad x\in[(\alpha-1)^{-1},\alpha(\alpha-1)^{-1}] \tag{19}$$

$\mathscr{R}_1$ and $\mathscr{R}_2$ define two functional transformations for coordinate changes $\hbar$ and both transformations are coupled under successive scale transformations. Let us remark that $\mathscr{R}_2$ is related to $\mathscr{R}_1$ by

$$\mathscr{R}_2[\hbar]=\mathscr{R}_1[\hbar\circ f_*^{-1}\circ\alpha^{-1}] \tag{20}$$

It is now possible to make connection with the result of ref. 1. Let us assume that $\hbar$ is an eigenvector in the sense that

$$\int_{-\sigma^2}^{\sigma} |h_*(v-(-\sigma)^{n-1})-h_*(v)|^q\, |\mathscr{R}_1[\hbar](h_*(v))|^q\, dv$$
$$\cong\sigma^{\tau(q)+1}\int_{-\sigma^2}^{\sigma} |h_*(v-(-\sigma)^{n-2})-h_*(v)|^q\, |\mathscr{R}_1[\hbar](h_*(v))|^q\, dv \tag{21}$$

and

$$\int_{-\sigma^2}^{\sigma} |h_*(u-(-\sigma)^n)-h_*(u)|^q\, |\hbar'(h_*(u))|^q\, du$$
$$=\sigma^{2\tau(q)+2}\int_{-\sigma^2}^{\sigma} |h_*(u-(-\sigma)^{n-2})-h_*(u)|^q\, |\hbar'(h_*(u))|^q\, du \tag{22}$$

Recalling that (12) is equal to the sum of (14) and (18) and that this holds for all values of n, we obtain

$$\sigma^{2\tau(q)}|\hbar'(h_*(u))|^q=\sigma^{\tau(q)}|\mathscr{R}_1[\hbar](h_*(u))|^q+|\mathscr{R}_2[\hbar](h_*(u))|^q \tag{23}$$

We now interpret Eq. (23) as an eigenvector equation in the space of coordinate changes which relate a map on the critical manifold to the fixed point f_*. This defines the "second" RG alluded to earlier. Clearly, the second RG is slaved to the first one, since it depends on h_*, or equivalently f_*^{-1}, whereas the first RG can be defined completely independently from the second. Because of definitions (15) and (19) for $\mathscr{R}_1$ and $\mathscr{R}_2$, the last equation is Eq. (3.33) of Kadanoff,[1] with the following correspondence between notations: $h_* \to \theta$, $\alpha^2 \circ f_*^{-1} \circ \alpha^{-2} \to \tilde{h}$, $|\hbar'|^q \to \psi$, $q \to -\tau$, $|\alpha|^q \sigma^{\tau(q)} \to \lambda$. Kadanoff has also given a numerical scheme to solve (23).

Here, we remark that the set of $\tau(q)$ can be found analytically within the linear approximation for the fixed point of the renormalization group. Actually, this corresponds to the first-order solutions of Kadanoff.[1] In this approach, f_* is approximated by a linear function [see Eqs. (6.9), (6.10) of ref. 3] on the interval $[\alpha^{-1}(1-\alpha)^{-1}, \alpha(\alpha-1)^{-1}]$ with a slope equal to $[\alpha(\alpha+1)]^{-1}$. In that case, the dominant eigenvector is simply the identity function $h(x) = x$, giving the following expression for $\tau(q)$:

$$\tau(q) = \frac{1}{\ln(\sigma)} \ln \left\{ \frac{1}{2} \sigma |\alpha|^{-q} (1 + \{1 + 4[\alpha(\alpha+1)]^q\}^{1/2}) \right\} - 1 \tag{24}$$

The numerical values obtained from Eq. (24) are reported in Table II of the preceding paper. Although the linear approximation gives a very accurate numerical estimate of $\tau(q)$, going beyond this approximation is worthwhile since, as we now discuss, it is then possible to make further analogies with other problems.

We emphasize that it is only in the linear approximation that the identity function on the domain of definition gives the leading scaling behavior for the multifractal moments. In the real case where the fixed-point function f_* is not approximated, each q in (23) associates to the qth multifractal moment a particular set of coordinate changes. As found by Kadanoff, the $q \to -\infty$ eigenvector can be found exactly from the fixed-point equation. In that limit, the RG for multifractals is given by $\mathscr{R}_2$ [cf. Eqs. (19) and (23)] and the eigenvectors are $\hbar(x) = (f_*^{-1} \circ \alpha^{-1})^p$ (p real). Since $\hbar$ is at least once differentiable, $p = 3$ corresponds to the dominant eigenvector and the multifractal exponents are asymptotically given by $\tau(q) = -3q \ln |\alpha| / \ln(\sigma)$. On the other hand, as q tends to infinity, $\mathscr{R}_1$ gives the most dominant contribution and the eigenvector is still the identity function $\hbar(x) = x$. In general, the leading scaling behavior as given by $\tau(q)$ comes from the most *dominant* eigenvalue of the second renormalization group equation (23). For each value of q, then, the set of all eigenvectors, *dominant* plus *nondominant*, paves the critical manifold of the golden-mean fixed point.

From the perspective of critical phenomena, these results may be seen as follows. The critical manifold of the usual first RG is a space of functions which are related to each other through once-differentiable coordinate changes $\hbar$ which preserve the cubic inflection point at the origin. These changes of coordinates constitute the gauge symmetry group of our problem. Since the second RG associates to each value of q a different eigencoordinate change $\hbar$ in function space, the multifractal moments, which are characterized by this second RG are in this sense gauge symmetry-breaking perturbations, since each value of q becomes associated with a particular coordinate change $\hbar$. Note also that we use the term "dominant" instead of "relevant" for $\tau(q)$ because the sign of $\tau(q)$ is not what determines wether the corresponding multifractal moment is asymptotically observable or not, since the sign of $\tau(q)$ can be arbitrarily changed by trivial rescaling of the distance at each iteration. All this is akin to multifractals in percolation, where each multifractal moment is associated with its own symmetry-breaking operator (in replica space)[6] and where there is also an arbitrary rescaling factor characterizing the multifractal properties.

4. UNIVERSAL RATIOS

Generally speaking, the existence of a renormalization group for observable quantities suggests that universality follows simply from a decomposition of these observables on the eigenvectors. Detailed proofs in the present case, however, do not appear straightforward. In this section, we demonstrate instead *heuristically* that the set of ratios derived in the previous paper are indeed universal. To proceed, we concentrate on the Fourier coefficients of the functions $M_q(F_n, h(u_1))$ defined in Eq. (5). At the critical point, these Fourier coefficients obey a fixed-point equation [see Eqs. (41) and (43)] reminiscent of the equation obeyed by the spectrum, and the scaling properties of these Fourier coefficients are corroborated by a numerical analysis. While relatively little can be shown for general q, in the special case $q = 1$ this Fourier analysis approach allows us to make a connection with the power spectrum which does have universal properties (see, e.g., refs. 3 and 4).

First, we discuss the problem of taking the average for the multifractal moments. Then, we shall consider their Fourier coefficients.

4.1. Invariant Measure and Multifractal Moments

As before, let h be the homeomorphism which conjugates the map to a pure rotation. Because we are interested in proving the universality of averages, we can consider $M_q(F_n, h(u_1))$ instead of $M_q(F_n, u_1)$ in the average over u_1. More precisely, it can be proved that averaging $M_q(F_n, h(u_1))$ over u_1 with respect to a uniform density gives the same result as averaging $M_q(F_n, u_1)$ with respect to the invariant measure associated with f. Below the critical line, this statement can be proved as follows.

Let f be a circle map. We call the invariant measure associated with f the probability density $P_f(x)$ which is such that

$$P_f(x) = P_f(f(x)) \left|\frac{df}{dx}\right| \tag{25}$$

on the domain of definition of f. Let R_σ, acting on the coordinate y, be related to f by a differentiable change of coordinate $x = h(y)$. In other words, we assume that

$$f \circ h = h \circ R_\sigma \tag{26}$$

The probability density P_{R_σ} associated with R_σ is then related to P_f through a change of coordinate

$$P_{R_\sigma}(y) = P_f(h(y))\, |dh(y)/dy| \tag{27}$$

Since the successive iterates of one point by a pure rotation with an irrational mean winding number cover uniformly the circle, the invariant measure associated with a pure rotation has a constant density. One can then check that $P_f(x)$ obtained by Eq. (27) with $P_{R_\sigma}(y)$ a constant indeed obeys Eq. (25) and is the invariant measure.

Our preceding statement is now proved since

$$\begin{aligned}\int_{-\sigma^2}^{\sigma} M_q(F_n, h(u_1))\, du_1 &= \int_{-\sigma^2}^{\sigma} M_q(F_n, h(u_1))\, P_{R_\sigma}(u_1)\, du_1 \\ &= \int_{-\sigma^2}^{\sigma} M_q(F_n, h(u_1))\, P_f(h(u_1))\, dh(u_1) \\ &= \int_{(\alpha-1)^{-1}}^{\alpha(\alpha-1)^{-1}} M_q(F_n, x)\, P_f(x)\, dx \end{aligned} \tag{28}$$

where we have discarded the absolute value, h being a strictly increasing function.

This discussion can be generalized to the case where the average of multifractal moments is performed with an arbitrary density and at criticality. Because our argumentation is more technical, it is reported in Appendix A.

4.2. Fourier Coefficients of the Multifractal Moments

Since, as shown in I and in Appendix A, the universal ratios correspond to averages over the invariant measure of f, it is more convenient to work directly with the function $M_q(F_n, h(u_1))$, whose Fourier coefficients are given by

$$\begin{aligned} a_q(k, F_n) &= \int_{-\sigma^2}^{\sigma} du\, e^{2\pi iku} M_q(F_n, h(u_1)) \\ &= \frac{1}{F_{n+1}} \sum_{1 \leqslant l \leqslant F_{n+1}} \int_{-\sigma^2}^{\sigma} du\, e^{2\pi iku} |h(\hat{R}^{(F_n)}(R^{(l)}(u))) - h(R^{(l)}(u))|^q \end{aligned} \tag{29}$$

Changing variable to $y = R^l(u)$ and using the periodicity of the integrand, we are left with

$$\begin{aligned} a_q(k, F_n) &= \frac{1}{F_{n+1}} \sum_{1 \leqslant l \leqslant F_{n+1}} e^{-2\pi ikl\sigma} \int_{-\sigma^2}^{\sigma} |h(u - (-\sigma)^n) - h(u)|^q e^{2\pi iku}\, du \\ &= \frac{1}{F_{n+1}} e^{-2\pi ik\sigma} \frac{1 - \exp(-2\pi i F_{n+1} \sigma k)}{1 - \exp(-2\pi ik\sigma)} \\ &\quad \int_{-\sigma^2}^{\sigma} |h(u - (-\sigma)^n) - h(u)|^q e^{2\pi iku}\, du \end{aligned} \tag{30}$$

Clearly, the universality of amplitude ratios is related to that of the Fourier coefficients. For example, we consider in the sequel the following quantities, which correspond to $A(q, 0; 2, 0)$ [defined in Eq. (16) of I]:

$$\frac{\langle\langle [M_q(F_n, x)]^2 \rangle\rangle}{\langle\langle M_q(F_n, x) \rangle\rangle^2} = \sum_{k \neq 0} \frac{|a_q(k, F_n)|^2}{a_q(0, F_n)^2} \tag{31}$$

and we show that its universal value follows from the fact that the low-frequency [$\omega = k\sigma \bmod(1) \ll 1$] Fourier coefficients a_k obey a universal scaling relation and that they give the dominant contribution to the ratios $A(q, 0; 2, 0)$ in the infinite-F_n limit.

The picture which will emerge from the analysis can be summarized as follows. By taking a frequency scale $\omega = k\sigma - m_k$, $0 \leqslant \omega < 1$, one defines a time scale which corresponds to a pure rotation with a rational winding number m_k/k. To take the limit ω goes to zero thus amounts to considering the periodic components of the multifractal moments with better and better approximations of the golden mean. These periodic components are at the scale of rational pure rotations and universality is recovered in the small-ω limit, but not in the general large-k limit. Actually, the situation is very similar to the problem of the frequency spectrum associated with maps on the critical manifold, where universality appears only in the $\omega \to 0$ limit. Both problems are indeed closely related, since it is shown in Section 4.2.2 that the Fourier coefficients of the $q=1$ moment can be expressed in terms of the spectrum of the map. For the corresponding $q=1$ amplitude ratios defined in I, we show that one gets rid of the nonuniversal corrections to scaling by taking the finite time scale F_n to infinity.

4.2.1. Arbitrary Value of q. Let us first consider heuristically the general q case. Restricting ourselves for the moment to the linear approximation of the fixed point, it is now shown that the Fourier coefficients have a scaling behavior.

For general q, let us define the fixed-point function $\tilde{S}_q^*(\omega, (-\sigma)^n)$ by

$$\tilde{S}_q^*(\omega, (-\sigma)^n) \equiv \int_{-\sigma^2}^{\sigma} e^{2\pi i k y} \, |h_*(y-(-\sigma)^n) - h_*(y)|^q \, dy \tag{32}$$

where the frequency ω, corresponding to an integer k, is defined as before as $\omega \equiv k\sigma \bmod(1)$ $(0 \leqslant \omega < 1)$ (hereafter, we shall use alternatively k or ω for convenience). By using the linear fixed point relation Eq. (6.12) of Rand *et al.*,[3] namely

$$h_*(\theta) = \begin{cases} \alpha^{-1} h_*(-\theta/\sigma), & -\sigma^2 < \theta \leqslant \sigma^3 \\ (1+\alpha^{-1})\, h_*(\theta/\sigma^2 - 1) + 1/(1-\alpha), & \sigma^3 \leqslant \theta \leqslant \sigma \end{cases}$$

we observe that

$$\begin{aligned} &\int_{-\sigma^2}^{\sigma} e^{2\pi i k y} \, |h_*(y-(-\sigma)^n) - h_*(y)|^q \, dy \\ &\quad = \sigma |\alpha|^{-q} \int_{-\sigma^2}^{\sigma} e^{-2\pi i \sigma k y} \, |h_*(y-(-\sigma)^{n-1}) - h_*(y)|^q \, dy \\ &\quad = \sigma^2 \, |1+\alpha^{-1}|^q \int_{-\sigma^2}^{\sigma} e^{2\pi i k \sigma^2 (y+1)} \, |h_*(y-(-\sigma)^{n-2}) - h_*(y)|^q \, dy \end{aligned} \tag{33}$$

By using arguments similar to those of Rand *et al.* [see ref. 3, Eq. (6.16)], we find that Eq. (33) shows that in the limit $\omega \to 0$, $\tilde{S}_q(\omega, (-\sigma)^n)$ obeys the same fixed-point equation as the one previously derived to find $\tau(q)$ in the linear approximation, namely

$$\tilde{S}_q^*(\omega, (-\sigma)^n) = \sigma \, |\alpha|^{-q} \tilde{S}_q^*(\omega/\sigma, (-\sigma)^{n-1}) + \sigma^2 \, |1+\alpha^{-1}|^q \, \tilde{S}_q^*(\omega/\sigma^2, (-\sigma)^{n-2}) \tag{34}$$

This then demonstrates that the fixed-point function $\tilde{S}_q^*(\omega, (-\sigma)^n)$ is a generalized homogeneous function of its arguments, so that

$$\tilde{S}_q^*(\omega, (-\sigma)^n) = \sigma^{(\tau(q)+1)} \tilde{S}_q^*(\omega/\sigma, (-\sigma)^{n-1}) \tag{35}$$

in the limit $\omega \to 0$.

Before proceeding with the Fourier coefficients (30), let us first organize, following ref. 3, all integer numbers k into bands according to the value of $\omega = k\sigma \bmod(1)$. For all positive or negative integers k, we may find a positive initeger j such that

$$\sigma^{j+1} \leqslant \omega = k\sigma \bmod(1) \leqslant \sigma^j \tag{36}$$

where the B_j band is defined as the interval $[\sigma^{j+1}, \sigma^j]$. Because σ is irrational, there exists a one-to-one correspondence between an integer k_j and a value of $\omega_j = k_j \sigma \bmod(1)$. Moreover, there exists a one-to-one correspondence between the bands: For each k_j of B_j, one can find a unique integer k_{j-1} of B_{j-1} such that

$$\omega_j = \sigma \omega_{j-1} \tag{37}$$

To prove this statement, recall that all ω are of the form $\omega = k\sigma - m$. By using $\sigma^2 = 1 - \sigma$, one demonstrates that the operation $\sigma(k\sigma - m_k) = -\sigma(m_k + k) + k$ defines a one-to-one relationship between two contiguous bands.

Since all frequencies are of the form $\sigma^l(r\sigma - s)$, with l, r, and s integers which are uniquely defined given a value of $k\sigma$, we use the following identities [recall that $F_n = F_{n-1} - (-\sigma)^n$]

$$\sigma^l F_n = (-1)^{n+1} \sigma^{n+l-1} \frac{1 - (-\sigma^{-2})^n}{1 + \sigma^{-2}}; \qquad l \geqslant n \tag{38a}$$

$$\sigma^l F_n = F_{n-l} + (-1)^{n+1} \sigma^{n+l-1} \frac{1 - (-\sigma^{-2})^l}{1 + \sigma^{-2}}; \qquad l \leqslant n \tag{38b}$$

to rewrite the Fourier coefficients in the l and n infinite limit as

$$a_q^*(\omega = \sigma^l(r\sigma - s), (-\sigma)^n)$$
$$\cong (-1)^{n+l}(1 + \sigma^{-2})^{-1} \frac{\sigma^{n-2l}}{F_{n+1}} \left[\frac{r\sigma^{-1} + s}{r\sigma - s} \right] \tilde{S}_q^*(\omega, (-\sigma)^n); \qquad l \leqslant n \tag{39}$$

So, in this limit, they obey the same scaling equation as the Fourier transform $\tilde{S}_q^*(\omega, (-\sigma)^n)$, namely

$$a_q^*(\omega, (-\sigma)^n) \cong \sigma^{(\tau(q)+1)} a_q^*(\omega/\sigma, (-\sigma)^{n-1}) \tag{40}$$

To check this scaling property, then, it suffices to concentrate on $\tilde{S}_q(\omega, (-\sigma)^n)$. Because of the analogy with the spectrum at the golden-mean rotation number, we further hypothesize that $\tilde{S}_q(\omega, (-\sigma)^n)$ is self-similar from one band B_j to another. In other words, we postulate that

$$\tilde{S}_q^*(\omega = \sigma^l(r\sigma - s), (-\sigma)^n) = \sigma^{n(\tau(q)+1)} \tilde{S}_q^*(\omega = r\sigma - s, 1) \tag{41}$$

In this general q case, we resort to numerical simulations to verify (41). As a consequence of unique ergodicity, we have

$$\begin{aligned} \tilde{S}_q(\omega, (-\sigma)^n) &= \int_{-\sigma^2}^{\sigma} e^{2\pi iky} \, |h(y - (-\sigma)^n) - h(y)|^q \, dy \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{0 \leqslant l \leqslant N-1} e^{2\pi ikR^l(0)} \, |h(\hat{R}^{(l+F_n)}(0)) - h(R^{(l)}(0))|^q \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{0 \leqslant l \leqslant N-1} e^{2\pi kil\sigma} \, |\hat{f}^{(l+F_n)}(0) - f^{(l)}(0)|^q \end{aligned} \tag{42}$$

which can be numerically evaluated for finite N. Figure 2 provides a check of the n dependence of Eq. (41) for two frequencies and various value of q through a plot of $\ln[\mathrm{Re}(\tilde{S}_q(\omega, (-\sigma)^n)]$ versus $\ln[\sigma^n]$. Figure 3, on the other hand, provides a verification of the l independence of Eq. (41), i.e., of the fact that it is *self-similar*, by plotting $\mathrm{Re}(\tilde{S}_q(\omega, (-\sigma)^n)/\sigma^{n\tau(q)})$ for $q=2$ and $n=17$. In the frequency domain considered, the principal peaks corresponding to σ^j, $j=5, 6,..., 9$, are well reproducible from one band to another. Other peaks are more sensitive to finite-size corrections corresponding to our relatively small value of n, but our numerical results support at least qualitatively the self-similar property of $\tilde{S}_q(\omega, (-\sigma)^n)$ and hence of the corresponding Fourier coefficients. Note that, by contrast, the scaling limit for the amplitude ratios is reached at much smaller values of n ($n \approx 8$ instead of $n \approx 17$).

To show that the set of $A(q, r; l, m)$ is universal, the mathematically correct procedure would be to project on a complete basis of eigenvectors of Eq. (23), and to retain the dominant eigenvector corresponding to each value of q. In this way, the Fourier coefficients should be of the form

$$a_q(\omega = \sigma^l(r\sigma - s), \sigma^n) \cong A_q \sigma^{n(\tau(q)+1)} a_q^*(\omega = r\sigma - s, 1) \tag{43}$$

where A_q is a nonuniversal constant corresponding to the projection onto the associated dominant eigenvector. Taking the ratios of the Fourier coefficients as in Eq. (31) eliminates the nonuniversal factor.

To verify that the Fourier coefficients are universal apart from a

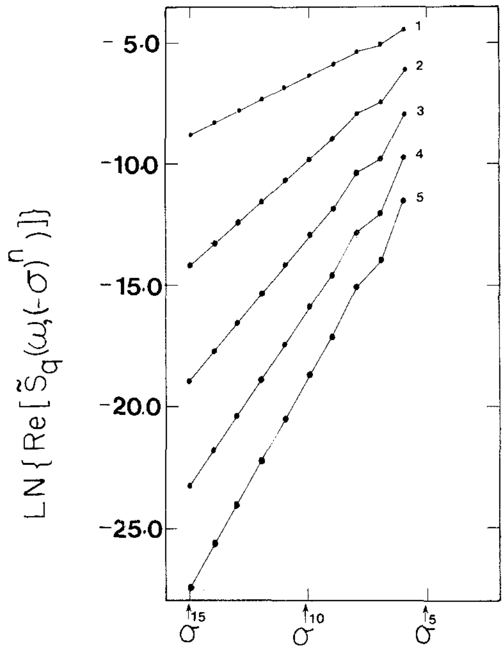

Fig. 2. Plot of $\ln\{\mathrm{Re}[\tilde{S}_q(\omega, (-\sigma)^n)]\}$ as a function of the finite-size time $\ln(\sigma^n)$. The numerical data have been obtained for two different frequencies, $\omega = \sigma^4$ and σ^5. On the scale of the figure both results are indistinguishable. Curves $i = 1,..., 5$ correspond to a multifractal moment of order $q = i$. In the scaling limit, $n \to \infty$, all curves tend to an asymptote whose slope s_q is given by the corresponding multifractal exponent $\tau(q) + 1$. In all cases, a numerical fit gives a value of $\tau(q)$ which compares well with the values listed in Table II of I ($s_1 = 0.66 \mp 0.01$; $s_2 = 1.81 \mp 0.01$; $s_3 = 2.49 \mp 0.02$; $s_4 = 3.07 \mp 0.02$; $s_5 = 3.61 \mp 0.02$).

normalization factor, Fig. 4 shows the ratio of $\tilde{S}_q(\omega, F_n)$ for the maps defined in Eqs. (1a) and (1b) of I. In the low-frequency limit, all the curves tend to constant, thereby corroborating Eq. (43).

4.2.2. $q=1$ Multifractal Moment and Spectrum of the Map. For $q = 1$, the Fourier coefficients can be expressed in terms of the spectrum corresponding to the map f. By definition, this observable $\tilde{f}(\omega)$ is given by[3,4] $[\omega = k\sigma \bmod(1)]$

$$\tilde{f}(\omega) \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{l=0}^{N-1} \exp(2\pi i k l \sigma) \, [f^{(l)}(0) - R^{(l)}(0)]$$

$$= \int_{-\sigma^2}^{\sigma} \exp(2\pi i k u) \, [h(u) - u] \, du \tag{44}$$

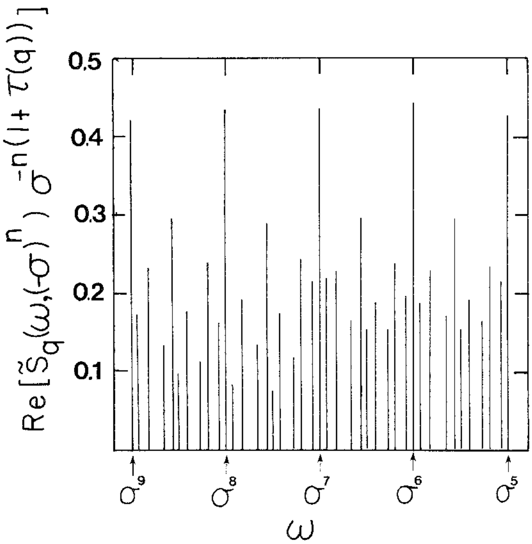

Fig. 3. Plot of $\mathrm{Re}[\tilde{S}_q(\omega, (-\sigma)^n)]\sigma^{-n(1+\tau(q))}$ as a function of ω for $q=2$. The principal peaks correspond to $n\omega=\sigma^m$, $m=5, 6,\ldots$. As explained in the text, all ω can be written as $\sigma^l(r\sigma-s)$, where $\sigma^2<r\sigma-s<\sigma$ and $l=1, 2,\ldots$. Bands are defined by the intervals $[\sigma^{l+1}, \sigma^l]$, $l=1,\ldots, \infty$. As in the case of the spectrum for maps of the cricle at the critical golden-mean rotation number, the function depicted here is self-similar from one band to another, i.e., is independent of l for any given set of frequencies which may be written in the form $\sigma^l(r\sigma-s)$. This result is explicitely demonstrated in the text in the case of $q=1$, where it is shown that $\tilde{S}_1$ can be written in terms of the spectrum. In the limit where ω goes to zero, the function $\tilde{S}_q(\omega, (-\sigma)^n)$ is universal apart from a nonuniversal multiplicative factor (see Fig. 4).

where definition (4) for the homeomorphism and unique ergodicity are used to derive the second line of (44).

From the work of Rand *et al.*, $\bar{f}(\omega)$ is *universal* in the $\omega\to 0$ limit and obeys the scaling relation (the subscript star is used to underline this universality)

$$\bar{f}_*(\omega) = -\sigma\bar{f}_*(\omega/\sigma), \qquad \omega \ll 1 \tag{45}$$

which is going to be related to Eq. (32) with $q=1$. Anticipating what follows, we note that the nonuniversal corrections to the scaling limit (45) are of order ω [i.e., $O(\omega)$].

To proceed, let us first express $\tilde{S}_1(\omega=\sigma^l(r\sigma-s), F_n)$ in terms of $\bar{f}(\omega)$. We first prove that $\tilde{S}_1(\omega=\sigma^l(r\sigma-s), F_n)$ obeys the scaling fixed-point equation (35) in the $\omega\to 0$ limit and then we prove that the ratios are

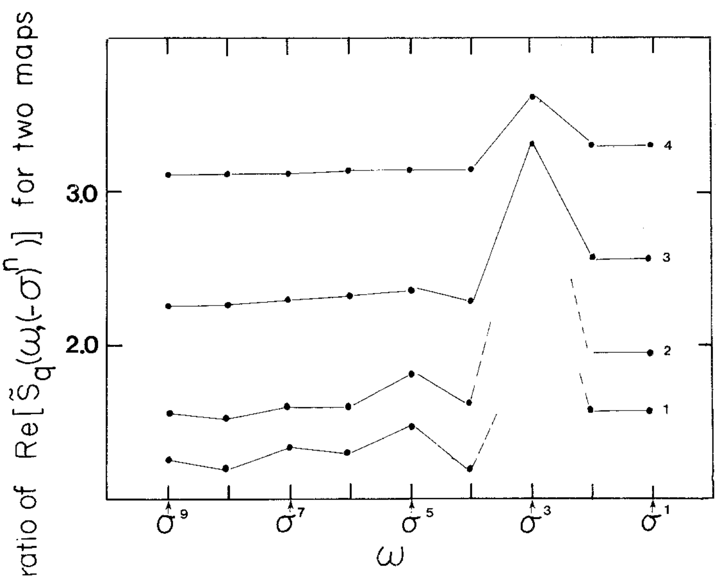

Fig. 4. Ratio of $\mathrm{Re}[\tilde{S}_q(\omega,(-\sigma)^n)]$ for the maps of Eqs. (1a) and (1b) of I for $q=2$ (curve 1), $q=3$ (curve 2), $q=4$ (curve 3), and $q=5$ (curve 4). In the low-frequency limit these ratios are independent of ω, corroborating thereby that the function $\tilde{S}_q(\omega,(-\sigma)^n)$ is universal, apart from a multiplicative constant, when ω goes to zero. In the case of $q=1$, this constant is equal to 1, and one recovers the case of the spectrum at the onset of chaos, where the universal character manifests itself in the low-frequency limit.

universal. For n odd, h being a strictly monotonically increasing function, we have

$$\begin{aligned}\int_{-\sigma^2}^{\sigma} & e^{2\pi iky}[h(y-(-\sigma)^n)-h(y)]\,dy \\ &= \int_{-\sigma^2}^{\sigma} e^{2\pi iky}[h(y-(-\sigma)^n)-(y-(-\sigma)^n)]\,dy \\ &\quad -\int_{-\sigma^2}^{\sigma} e^{2\pi iky}[h(y)-y]\,dy-(-\sigma)^n\int_{-\sigma^2}^{\sigma} e^{2\pi iky}\,dy \qquad (46)\end{aligned}$$

Changing to the variable $y=u-(-\sigma)^n$ and using $(-\sigma)^n=F_{n-1}-\sigma F_n$ in the first integral leads to the following relationship with the spectrum $[\omega\equiv k\sigma \bmod(1)]$:

$$\tilde{S}_1(\omega,(-\sigma)^n)=(-1)^{n \bmod(2)}[1-\exp(-2\pi i\omega F_n)]\,\tilde{J}(\omega) \qquad (47)$$

We have thus demonstrated that the nontrivial functional dependence of the Fourier coefficients of the first multifractal moment is given by the spectrum. Writing $\omega = \sigma^l(r\sigma - s)$, we have, in the limit where $1 \ll l \ll n$,

$$\tilde{S}_1(\omega = \sigma^l(r\sigma - s), F_n)$$
$$= -2\pi i \frac{(-1)^l}{1+\sigma^{-2}} \left[\frac{r}{\sigma} + s\right] \sigma^{n-l-1} \tilde{f}(\omega = \sigma^l(r\sigma - s)) \tag{48}$$

As l tends to infinity, $\tilde{f}$ tends to the universal functional $\tilde{f}_*$ and, because of Eq. (45),

$$(-\sigma)^{-l} f(\omega = \sigma^l(r\sigma - s))$$

tends to an l-independent *universal constant* given by

$$(-\sigma)^{-l} \tilde{f}_*(\omega = \sigma^l(r\sigma - s))$$

Recalling that $\tau(1) = 0$, this proves that $\tilde{S}_1(\omega = \sigma^l(r\sigma - s), F_n)$ obeys the fixed-point equation (35).

To prove the universal character of the ratio defined by Eq. (31), we now proceed in close parallel with the reasoning of Rand *et al.* for the spectrum. It is shown that the low-frequency peaks give the dominant contribution to the $q = 1$ moment. As discussed above, its universal character will follow.

Let us for the moment neglect the nonuniversal terms of order ω. This statement is equivalent to replacing $\tilde{f}(\omega)$ by its low-frequency universal limit $\tilde{f}_*(\omega)$, which also obeys the following self-similarity relation at the golden-mean winding number:

$$\tilde{f}_*(\omega) = -\sigma \tilde{f}_*(\omega/\sigma) \tag{49}$$

for all ω in $[0, 1]$. It will be demonstrated hereafter that the $O(\omega)$ terms neglected in Eq. (49) do not contribute in the infinite-F_n limit.

As a next step, the sum (31) can be rewritten by considering all integers k_0 of the $B_{j=0}$ band. Making use of the correspondence between the bands, we now express Eq. (31) as $[\omega_0 \equiv k_0 \sigma \bmod(1)]$

$$|a_1(0, F_n)|^2 \sum_{k_0 \in B_0} \left[\sum_{l \geq 0} |a_1(\omega_0 \sigma^l, F_n)|^2\right] \tag{50}$$

and, by using Eqs. (30) and (47), we find that each k_0 term contributes to the sum as

$$\sum_{l \geq 0} \left|(1+\sigma^2) \frac{1 - \exp(-2\pi i F_{n+1} \omega_0 \sigma^l)}{1 - \exp(-2\pi i \omega_0 \sigma^l)} [1 - \exp(-2\pi i \omega_0 \sigma^l F_n)]\, \omega_0 \sigma^l\right|^2$$
$$\times |\omega_0^{-1} \sigma^{-l} \tilde{f}_*(\omega_0 \sigma^l)|^2 \tag{51}$$

Finally, the exact self-similarity of the spectra allows us to replace the last factor by its l-infinite limit, which depends on ω_0, but not on l. The universal ratio $A(q, 0; 2, 0)$ is obtained by summing over all ω_0 of the B_0 band.

Although the convergence of the series given by Eq. (51) can be proved, since the general term scales with n as

$$|a(\omega_0\sigma^l, F_n)|^2/|a(0, F_n)|^2 \sim \sigma^{4|n-l|}|\omega_0^{-1}\sigma^{-l}\mathcal{J}_*(\omega_0\sigma^l)|^2 \tag{52}$$

for $l \gg n \gg 1$ and $1 \ll l \ll n$, a numerical analysis shows that this series defines a highly fluctuating function of ω_0 which cannot be approximated by a polynomial in ω_0. The scalling behavior of Eq. (52) can, however, be useful to understand why nonuniversal corrections are eliminated in the n-infinite limit. From (6.7) of ref. 4, we write that the spectrum is universal with corrections that vary as ω,

$$\omega_0^{-1}\sigma^{-l}\mathcal{J}(\omega_0\sigma^l) = \omega_0^{-1}\sigma^{-l}\mathcal{J}_*(\omega_0\sigma^l)[1 + O(\omega_0\sigma^l)] \tag{53}$$

As a consequence, these corrections contribute to the universal ratio by an amount of order σ^n and can thus be discarded when n goes to infinity.

Finally, we note that the above results have answered two kinds of questions. First, we have demonstrated that the fluctuations of the multifractal moments are scale independent in the sense that the finite time scale F_n enters the Fourier coefficients only as a power of $\sigma^{\tau(q)+1}$, as can be seen from Eq. (41) and the arguments that followed. This demonstrates also that the successive powers of any multifractal moment scale with a trivial exponent given by a linear function of $\tau(q)$. This is the gap scaling property analyzed in I. Second, it has been shown that the universal character of our multifractal analysis comes from the low-frequency part of the Fourier coefficients $[k\sigma \bmod(1) \ll 1]$ and not from the infinite-k limit, in complete analogy with the case of the spectrum.

5. CONCLUSION

In the first part of this paper, a simplified version of the renormalization group analysis of Kadanoff[1] for the multifractal properties of maps of the circle has been presented. From Eqs. (21)–(23), this transformation acts on the space of coordinate changes which relate a map on the golden-mean critical manifold to the fixed point of the usual RG for maps of the circle.[3–5] This RG is a "second" RG in the sense that it depends on the fixed point of the usual RG for maps of the circle, whereas the latter can be defined without reference to the former. While the usual RG does not, by itself, give the set of $\tau(q)$ as eigenvalues, the derivation shows that all that the second RG does is to extract the multifractal information on the critical manifold from the fixed point of the usual RG.

In the case where the fixed-point function of Rand *et al.*[3] is not approximated by a piecewise linear function, the eigenvalue equations which given the multifractal exponents break a symmetry property reminiscent of a global gauge symmetry. More specifically, although the critical manifold is globally invariant under all changes of coordinate preserving the cubic inflection point, the multifractal moments select on this critical manifold different eigendirections according to the order of the multifractal moment considered. To each multifractal moment, then, is associated a "dominant" (largest) eigenvalue from which $\tau(q)$ is extracted. The other eigenvalues are "nondominant." From our point of view, the characterization of operators as relevant or irrelevant is ill adapted to the multifractal analysis, for two reasons. First, a symmetry-relevant eigenvalue usually means the fixed point considered is unstable toward other fixed points with different symmetries. Such fixed points cannot be identified here. Second, a trivial length rescaling at each iteration can shift the $\tau(q)$ from positive to negative, and this cannot generally be done in a standard RG analysis.[6] The situation depicted here is analogous to the case encountered in percolation.[6]

In the spirit of I, we have also, in Section 4, analyzed the multifractal moments as functions of the starting point of the series which define them. It was shown that the Fourier coefficients of these functions obey a fixed-point equation which gives $\sigma^{\tau(q)+1}$ as eigenvalues. This has demonstrated the gap scaling of the previous paper, or in other words, the scale-independence property of the normalized fluctuations of the multifractal moments (as a function of the starting point). As in the case of the spectrum at the onset of chaos, we note that universality is recovered in the low-frequency limit, where the system is probed at time scales which approximate the golden mean. In the general case, this characteristic property has been numerically corroborated and we have shown that the Fourier coefficients of any multifractal moment can be written in a scaling form [cf. Eq. (40) or (41)] reminiscent of that obeyed by the spectrum at the onset of chaos, where not only universality, but also self-similarity, are recovered at the critical golden-mean rotation number. For the dependence of the first multifractal moment on the starting point, the analysis has been

[5] For a mathematical study of the invariant measure at the critical point, see ref. 12.

[6] A notable exception is the Gaussian fixed point for a ϕ^4 theory, but, in that case, relevant eigenoperators become *dangerously* irrelevant and this is not the case in the context of maps of the circle. For more discussion of multifractals in the context of standard critical phenomena, see ref. 13.

analytically pushed further, emphasizing the connection between multifractals and the spectrum at criticality, both problems being essentially the same in this case [cf. Eq. (48)]. We have, moreover, demonstrated in Appendix A that the universal ratios defined in I are independent of the probability distribution with which the starting point is drawn. This means that our statistical point of view is consistent and can be carried out in practice.

Finally, since the dependence of the multifractal moments on the starting point is characterized by the analog of universal spectra, and in particular by the usual spectrum in the case of the $q=1$ moment, our statistical approach to this dependence on the starting point can be considered as an alternative route to access the universal properties of the spectrum. We emphasize that the scaling regime for the universal ratios is much more easily accessible than for the spectrum, a point which should be important for experiments.

APPENDIX A. AVERAGED MULTIFRACTAL MOMENTS DO NOT DEPEND ON THE *A PRIORI* PROBABILITIES

Let us first recall Eq. (11a) of paper I,

$$M_q(F_n, x_1) = M_q(F_n, f(x_1)) + \frac{1}{F_{n+1}} \{ |\hat{f}^{(F_n)}(x_1) - x_1|^q - |\hat{f}^{(F_n)}(x_{F_{n+1}+1}) - x_{F_{n+1}+1}|^q \} \tag{A1}$$

We know that

$$|\hat{f}^{(F_n)}(x_1) - x_1|^q < |\hat{f}^{(F_n)}(0) - 0|^q \qquad \text{for all} \quad x_1$$

The scaling near the origin is also known since the work of Shenker: $|\hat{f}^{(F_n)}(0)| \cong c/|\alpha|^n$, where c is a positive constant. Hence, we can write that

$$\frac{1}{F_{n+1}} |\hat{f}^{(F_n)}(x_1) - x_1|^q \leqslant c F_{n+1}^{-\tau_\infty(q)-1} \tag{A2}$$

where

$$\tau_\infty(q) \equiv -\frac{\ln|\alpha|}{\ln \sigma} q \tag{A3}$$

Since the last two terms in the bracket in (A1) are bounded by the same quantity, the absolute value of their difference is bounded by twice the right-hand side of (A2). Defining $\mathscr{A}_q(F_n, x_1) \equiv M_q(F_n, x_1)/F_{n+1}^{-\tau(q)-1}$, we can thus write

$$\mathscr{A}_q(F_n, x_1) = \mathscr{A}_q(F_n, f(x_1)) + E(F_n, x_1) \tag{A4}$$

where

$$|E(F_n, x_1)| < 2cF_{n+1}^{\tau(q)-\tau_\infty(q)} \tag{A5}$$

Denoting the right-hand side of this equation by E for short, and supposing that the starting point x_1 is sampled with an *a priori* probability $\mathscr{P}$, we can find an upper bound to the average of $\mathscr{A}_q(F_n, x_1)$ over the starting point x_1,

$$\langle \mathscr{A}_q \rangle = \int_0^1 \mathscr{A}_q(F_n, x_1)\, d\mathscr{P}\{x_1\} \tag{A6}$$

namely,

$$\begin{aligned} &\leqslant \int_0^1 \frac{1}{L} \{\mathscr{A}_q(F_n, x_1) + [\mathscr{A}_q(F_n, x_2) + E] + \cdots \\ &\quad + [(\mathscr{A}_q(F_n, x_L) + (L-1)E]\}\, d\mathscr{P}\{x_1\} \end{aligned} \tag{A7}$$

This upper bound is equal to

$$\int_0^1 \frac{1}{L} \{\mathscr{A}_q(F_n, x_1) + \mathscr{A}_q(F_n, x_2) + \cdots + \mathscr{A}_q(F_n, x_L)\}\, d\mathscr{P}\{x_1\} + \frac{1}{2}(L-1)E \tag{A8}$$

The rest, $(L-1)E/2$, is equal to

$$(L-1)cF_{n+1}^{\tau(q)-\tau_\infty(q)} \tag{A9}$$

which, because $\tau(q) - \tau_\infty(q) < 0$, vanishes for any arbitrarily large L, as long as n is large enough (n goes to infinity before L). A lower bound to the average moment (A6) is the integral of the sum, minus $(L-1)E/2$ as rest, instead of plus as in (A8). Clearly, then, in the limit of large enough n, the average multifractal moment (A6) becomes equal to

$$\int_0^1 \frac{1}{L} \{\mathscr{A}_q(F_n, x_1) + \mathscr{A}_q(F_n, x_2) + \cdots + \mathscr{A}_q(F_n, x_L)\}\, d\mathscr{P}\{x_1\} \tag{A10}$$

That integral can be rewritten by using changes of variables $x_1 = \psi^{(i)}(x_i)$ with $\psi \equiv f^{-1}$; $\psi^{(k)} \equiv \psi \circ \psi \circ \psi \circ \psi \cdots \circ \psi$; k times. The problem of the independence with respect of the *a priori* probability is then rephrased in terms of the convergence for probability laws

$$\lim_{L \to \infty} \frac{1}{L} \{\mathscr{P}(x) + \mathscr{P}(\psi(x)) + \mathscr{P}(\psi^{(2)}(x)) \cdots + \mathscr{P}(\psi^{(L)}(x))\} \tag{A11}$$

for an arbitrary distribution of the starting point. The proof of the convergence of this equation to the *invariant measure* for almost all $\mathscr{P}$ can be found in ref. 11. Then the average multifractal moment (A6) is independent of $\mathscr{P}$, a result that can be checked numerically (see Table I of I). By a similar argument, the moments of the $\mathscr{A}_q$, and by extension the probability distribution of the multifractal moments itself, are independent of the *a priori* probability distribution with which the starting point is drawn.

APPENDIX B. DERIVATION OF EQS. (1) AND (5)

Equation (5b) establishes a correspondence between the problem of multifractal moments and that of differentiability of the conjugacy to a pure rotation. It is this correspondence which allows some analytic work, and, as we see below, the connection is most natural when one works with multifractal moments which are averaged over the starting point.

First, let us prove[7] Eq. (1). For this, it suffices to apply g_n to $f(x)$. In this way, one finds

$$\begin{aligned} g_n \circ f(x) &= \frac{1}{n} \sum_{k=1}^{n} [f^{(k+1)}(x) - k\rho] \\ &= \frac{1}{n} \sum_{k=2}^{n+1} [f^{(k)}(x) - (k-1)\rho] \\ &= g_n(x) + \rho + \left\{\frac{1}{n} [f^{(n+1)}(x) - f(x)] - \rho\right\} \end{aligned} \tag{B1}$$

But, by definition of a pure rotation, one has

$$R_\rho(g_n(x)) = g_n(x) + \rho \tag{B2}$$

and by definition of the mean rotation number[8]

$$\lim_{n \to \infty} \frac{1}{n} [f^{(n)}(y) - y] = \rho \tag{B3}$$

which is independent of x. QED

The main statement which we wish to prove in this Appendix is that the closest-return distances *averaged* upon the starting point are related to the conjugacy homeomorphism through

$$\left\langle \frac{1}{F_{n+1}} \sum_{i=1}^{F_{n+1}} |\hat{f}^{(F_n)}(x_i) - x_i|^q \right\rangle = \int_0^1 |h(u-(-\sigma)^n) - h(u)|^q \, du \qquad \text{(B4)}$$

where by definition $x_i \equiv f^{(i-1)}(x_1)$, $1 \leqslant i \leqslant F_{n+1}$, $\hat{f}^{(F_n)} \equiv f^{(F_n)} - F_{n-1}$, and $h \circ R_\sigma = f \circ h$. The following proof is valid when the conjugacy $h = g^{-1}$ is differentiable. At the critical point, where this is not the case, the statement (B4) is justified *a posteriori* by the approximate RG results obtained in (24), which are in good agreement with the numerical results.

To simplify the proof, first relabel the points x_1 to $f^{(F_{n+1}+F_n-1)}(x_1)$ appearing in the sum in (B4) as follows: Let $\tilde{x}_1 \equiv f^{(F_{n+1}-1)}(x_1)$ and $\tilde{x}_i \equiv f^{(-i+1)}(\tilde{x}_1)$ with $1 \leqslant i \leqslant F_{n+1}$. Because the mean values of the moments are independent of the *a priori* probability distribution for the starting point (see Appendix A) and are therefore independent of whether the average is taken over the first of or any other point of the iteration, let us average with a uniform probability distribution over the point $f(\tilde{x}_1)$. One then finds

$$\begin{aligned}
&\frac{1}{F_{n+1}} \sum_{i=1}^{F_{n+1}} \int_0^1 |\hat{f}^{(F_n)}(\tilde{x}_i) - \tilde{x}_i|^q \, d(f(\tilde{x}_1)) \\
&= \frac{1}{F_{n+1}} \sum_{i=1}^{F_{n+1}} \int_0^1 |\hat{f}^{(F_n)}(\tilde{x}_i) - \tilde{x}_i|^q \left[\frac{df(\tilde{x}_1)}{d\tilde{x}_i} \right] d\tilde{x}_i \\
&= \frac{1}{F_{n+1}} \sum_{i=1}^{F_{n+1}} \int_0^1 |\hat{f}^{(F_n)}(\tilde{x}_i) - \tilde{x}_i|^q \left[\frac{d(f^{(i)}(\tilde{x}_i))}{d\tilde{x}_i} \right] d\tilde{x}_i \\
&= \frac{1}{F_{n+1}} \sum_{i=1}^{F_{n+1}} \int_0^1 |\hat{f}^{(F_n)}(x) - x|^q \left[\frac{d(f^{(i)}(x))}{dx} \right] dx \\
&= \int_0^1 |\hat{f}^{(F_n)}(x) - x|^q \left\{ \frac{1}{F_{n+1}} \sum_{i=1}^{F_{n+1}} \frac{d(f^{(i)}(x))}{dx} \right\} dx \\
&= \int_0^1 |\hat{f}^{(F_n)}(x) - x|^q \, dg_{F_{n+1}}(x)
\end{aligned} \qquad \text{(B5)}$$

where we have used Eq. (1) for the definition of $g_{F_{n+1}}(x)$. For finite n, $g_{F_{n+1}}(x)$ is everywhere differentiable. One can then change variable to $x = g_{F_{n+1}}^{-1}(u)$ in the last integral to find

$$\int_0^1 |\hat{f}^{(F_n)}(g_{F_{n+1}}^{-1}(u)) - g_{F_{n+1}}^{-1}(u)|^q \, du \qquad \text{(B6)}$$

By definition of conjugacy to a rotation, one has

$$\begin{aligned}\hat{f}^{(F_n)}(g^{-1}_{F_{n+1}}(u)) &= f^{(F_n)}(g^{-1}_{F_{n+1}}(u)) - F_{n-1}\\ &= f^{(F_n)} \circ g^{-1} \circ g \circ g^{-1}_{F_{n+1}}(u) - F_{n-1}\\ &= g^{-1} \circ R^{(F_n)}_{\sigma} \circ g \circ g^{-1}_{F_{n+1}}(u) - F_{n-1}\\ &= g^{-1} \circ [g \circ g^{-1}_{F_{n+1}}(u) - (-\sigma)^n]\end{aligned}$$

because

$$R^{(F_n)}_{\sigma}(X) = X + F_n\sigma = X + F_{n-1} - (-\sigma)^n \tag{B7}$$

As n tends to infinity, $(g_{F_{n+1}})^{-1}$ tend to $g^{-1,}$ so that one can write

$$g \circ g^{-1}_{F_{n+1}}(u) = u + \varepsilon_n(u) \tag{B8}$$

where $\varepsilon_n(u)$ tends to a function identically equal to zero as n goes to infinity. We then have that

$$\begin{aligned}&g^{-1} \circ [g \circ g^{-1}_{F_{n+1}}(u) - (-\sigma)^n] - g^{-1}_{F_{n+1}}(u)\\ &\quad = g^{-1} \circ [g \circ g^{-1}_{F_{n+1}}(u) - (-\sigma)^n] - g^{-1} \circ [g \circ g^{-1}_{F_{n+1}}(u)]\end{aligned} \tag{B9}$$

can be approximated by

$$g^{-1}(u - (-\sigma)^n) - g^{-1}(u) + \text{terms smaller by a factor of order } \varepsilon_n(u) \tag{B10}$$

when g^{-1} can be expanded in Taylor series. After replacing the first-order approximation (B10) in Eq. (B6), and using $g^{-1} \equiv h$, one has the result of Eq. (B4).

An alternate proof of (B4) follows the steps of Eqs. (25)–(29) by considering the invariant measure associated with the map f. In Eq. (B5), $dg_{F_{n+1}}$ approximates this invariant measure [see Eq. (27)] and by changing the argument in (B5) to $x \to h(x)$, one gets directly the desired result (B4).

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